

FINITE TIME BLOWUP OF THE STOCHASTIC SHADOW GIERER-MEINHARDT SYSTEM

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ABSTRACT. By choosing some special (random) initial data, we prove that with probability 1, the stochastic shadow Gierer-Meinhardt system blows up pointwisely in finite time. We also give a (random) upper bound for the blowup time and some estimates about this bound. By increasing the amplitude of the initial data, we can get the blowup in any short time with positive probability.

Keywords: Stochastic shadow Gierer-Meinhardt system, Finite time blowup, Brownian motions, Itô formula.

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1. INTRODUCTION

Many of the mathematical models that have been proposed for the study of population dynamics, biochemistry, morphogenesis and other fields, take the following form:

$$(1.1) \quad \begin{cases} \partial_t u = d_1 \Delta u + f(u, v) & \text{in } \mathcal{O} \times (0, T), \\ \tau \partial_t v = d_2 \Delta v + g(u, v) & \text{in } \mathcal{O} \times (0, T), \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial \mathcal{O} \times (0, T), \end{cases}$$

where $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is Laplace operator, \mathcal{O} is a bounded smooth domain in \mathbb{R}^n with unit outward normal vector ν on its boundary $\partial \mathcal{O}$; the two positive constants d_1, d_2 are the diffusion rates of two substances u and v respectively; $\tau > 0$ is the number tuning response rate of v related to the change of u ; f, g are both smooth functions referred to as the reaction terms.

As we choose

$$(1.2) \quad f(u, v) = \frac{u^p}{v^q}, \quad g(u, v) = \frac{u^r}{v^s},$$

with $p > 0, q > 0, r > 0, s \geq 0$ satisfying the condition:

$$(1.3) \quad 0 < \frac{p-1}{r} < \frac{q}{s+1},$$

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Eq. (1.1) is the well known Gierer-Meinhardt system. When Δ 's are removed, the corresponding ODEs have a stable equilibrium solution $(1, 1)$. The condition (1.3) is imposed so that $(1, 1)$ becomes *unstable* due to the two diffusion terms with d_1 small and d_2 large. This idea was proposed by Turing in 1952 and used to explain the onset of pattern formation by an instability of an unpatterned state leading to a pattern. It is now commonly called *Turing diffusion-driven instability* ([25]). Based on this idea, the Gierer-Meinhardt system (1.1)-(1.3) was formulated in 1972 [7] to model the regeneration phenomena of *hydra*.

The Gierer-Meinhardt system (1.1)-(1.3) is usually called *full* system, its dynamics remains far from being understood at this time. First result in this direction was due to Rothe in 1984 [24], but only for a very special case $n = 3$, $p = 2$, $q = 1$, $r = 2$ and $s = 0$. In 1987, a result for a related system was obtained in [18]. The nearly optimal resolution for the global existence issue came in 2006 with an elementary and elegant proof by Jiang [10]. In [10], the global existence was established for the range $\frac{p-1}{r} < 1$. This only leaves the critical case $\frac{p-1}{r} = 1$ still open, since it has been known already that in case $\frac{p-1}{r} > 1$ blow-up can occur even for the corresponding kinetic system ([21]).

When $d_2 \rightarrow \infty$, we expect that v tends to be space-homogeneous, i.e., $v(x, t)$ will be a spatially constant but time dependent function $\xi(t)$. Now the above Gierer-Meinhardt system is replaced by the following *shadow system*:

$$(1.4) \quad \begin{cases} \partial_t u = d_1 \Delta u - u + \frac{u^p}{\xi^q} & \text{in } \mathcal{O} \times (0, T), \\ \tau \frac{d\xi}{dt} = \left(-\xi + \frac{\overline{u^r}}{\xi^s} \right) & \text{in } (0, T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathcal{O} \times (0, T), \end{cases}$$

where $\overline{u^r} = \frac{1}{|\mathcal{O}|} \int_{\mathcal{O}} u^r dx$ with $|\mathcal{O}|$ being the volume of \mathcal{O} . This idea was suggested by Keener ([12]) and the name "shadow system" was proposed by Nishiura ([22]).

The dynamics (1.4) has been less well studied than the full Gierer-Meinhardt system. Global existence and finite-time blow-up have firstly been explored by the first author and Ni ([15]) in 2009. In particular, they show that for $\frac{p-1}{r} < \frac{2}{n+2}$ there is a unique global solution, whereas for $\frac{p-1}{r} > \frac{2}{n}$ blow-up can occur provided that $p = r$, $\tau = s + 1 - q$. Later, Phan showed that Eq. (1.4) also admits a global solution in the case $\frac{p-1}{r} = \frac{2}{n+2}$ ([23]). The first author and Yip continue the work in [15] and improve the earlier results concerning blowup solutions to the optimal case $\frac{p-1}{r} > \frac{2}{n+2}$.

Since the existence and blowup of the solutions do not depend on the numbers d_1 and τ . Without loss of generality, we shall assume $d_1 = 1$ and $\tau = 1$ throughout the rest of this paper.

The purpose of this paper is to study the shadow Gierer-Meinhardt system with random migrations with the following form:

$$(1.5) \quad \begin{cases} \partial_t u = \Delta u - u + \frac{u^p}{\xi^q} & \text{in } \mathcal{O} \times (0, T), \\ d\xi = \left(-\xi + \frac{\overline{u^r}}{\xi^s}\right) dt + \xi dB_t & \text{in } (0, T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{O} \times (0, T), \\ u(0) = u_0 & \text{in } \mathcal{O}, \\ \xi(0) = \xi_0, \end{cases}$$

where ξdB_t can be explained as random migrations and B_t is a one-dimensional standard Brownian motion. Due to the random effects, we need to introduce the sample space Ω and re-define

$$u(t, x, \omega) : \mathbb{R}^+ \times \mathcal{O} \times \Omega \rightarrow \mathbb{R}^+, \quad \xi(t, \omega) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+ \setminus \{0\}.$$

To our knowledge, there seem only two papers in the research of stochastic Gierer-Meinhardt type systems. One is [13], which studied a system including two coupled stochastic PDEs with bounded and Lipschitz nonlinearity. [13] only proved the *local* existence of the *positive* stochastic solution by Da Prato-Zabczyk's approach ([5]). The other is [27] established the *global* existence of the strong positive solution and the large deviation principle for Eq. (1.5).

We shall study in this paper the blowup problem of Eq. (1.5) under quite general assumptions. When $p \geq r$ and $\frac{p-1}{r} > \frac{2}{n+2}$, we show that with probability 1, Eq. (1.5) blows up pointwisely if we choose some suitable (random) initial data. We also give a (random) upper bound for the blow up time and consequently obtain a probabilistic estimate of this blow up.

To our knowledge, there are not many results for the blow up of stochastic systems. The work [1] proved that the 2nd moment of the solution of some nonlinear wave equations blow up, while [2] gave a nice criterion for the blow up of some stochastic reaction-diffusion equations under p th moments. As pointed out in [2], the blowup under p th moments even does not imply the pathwise blowup with a positive probability. [4] extended the result in [2] to the case of stochastic parabolic equations with delay. Most recently, Chow and Khasminski established an almost sure blowup result for a family of SDEs ([3]). [20] and [19] studied stochastic heat equations and showed that the noises can produce blowup with positive probability. In contrast, our blowup results depend on the choices of initial data, it is inspired by the deterministic work of [8], [16] and [15]. A special (random) data can, with probability 1, lead to a blowup of the SPDEs solutions. By increasing the amplitude of the initial data, we can get the blowup in any short time with positive probability.

Both probabilistic and PDE's methods play important roles in our approach. Itô formula in the proof of Lemma 2.2 below is the key point for finding the monotone stochastic process $\hat{\xi}(t)$, which paves the way to applying classical PDE techniques and estimating the upper bounds of blow up time. For the PDE's argument, we follow the approaches shown in [15] and [16].

The organization of the paper is as follows. In section 2, we introduce some notations and give some prerequisite lemmas. To show our approach more transparently, we prove a blowup theorem under some additional assumption in section 3. The 4th section removes the assumption and build the general blowup result by integral estimates.

2. SOME AUXILIARY LEMMAS AND A MONOTONE STOCHASTIC PROCESS $\hat{\xi}(t)$

From now on, we assume $\mathcal{O} = B_1(0)$, the unit open ball in \mathbb{R}^n with zero center. For the notational simplicity, write $v(t, z) = e^t u(t, z)$ for all $t > 0$ and $z \in \bar{B}_1(0)$ and

$$(2.1) \quad K(t) = \frac{e^{-(p-1)t}}{\xi^q(t)},$$

then

$$(2.2) \quad \begin{cases} \partial_t v = \Delta v + K(t)v^p & \text{in } B_1(0), \\ d\xi = \left(-\xi + e^{-rt} \frac{v^r}{\xi^s}\right) dt + \xi dB_t, \\ \frac{\partial v}{\partial \nu} = 0 \\ v(0) = u_0, \\ \xi(0) = \xi_0, \end{cases} \quad \text{on } \{z = 1\},$$

To study the blow up of Eq. (1.5), we only need to study that of Eq. (2.2). So we shall concentrate on the blow up of v and ξ in the sequel.

Write

$$B_t^* = \sup_{0 \leq s \leq t} |B_s| \quad \forall t > 0,$$

it is well known ([11, p. 96]) that for any $A > 0$,

$$\mathbb{P}(B_t^* \geq A) \leq \frac{\sqrt{t}}{\sqrt{2\pi}} \frac{4}{A} e^{-\frac{A^2}{2t}} \quad \forall t > 0.$$

Hence,

$$\mathbb{P}(B_t^* < \infty) = 1 - \lim_{A \rightarrow \infty} \mathbb{P}(B_t^* \geq A) = 1 \quad \forall t > 0.$$

For every $t > 0$, denote $\mathcal{N}_t = \{\omega : B_t^* = \infty\}$, it is clear that $\mathbb{P}(\mathcal{N}_t) = 0$. Take $t = 1, 2, \dots$, it is easy to see that $\mathcal{N}_t \subset \mathcal{N}_m$ for all $t \leq m$. Define $\mathcal{N} = \lim_{m \rightarrow \infty} \mathcal{N}_m$, we have $\mathbb{P}(\mathcal{N}) = \lim_{m \rightarrow \infty} \mathbb{P}(\mathcal{N}_m) = 0$. Hence, for all $\omega \in \Omega \setminus \mathcal{N}$,

$$B_t^*(\omega) < \infty \quad \forall t > 0.$$

From the above observation, without loss of generality, we can assume that for all $\omega \in \Omega$,

$$(2.3) \quad B_t^*(\omega) < \infty \quad \forall t > 0.$$

For all $x \in \mathbb{R}^n$, denote $z = |x|$. Consider the following isotropic function

$$\phi(z) = \begin{cases} z^{-\alpha}, & \delta \leq z \leq 1, \\ \delta^{-\alpha}(1 + \frac{\alpha}{2}) - \frac{\alpha}{2}\delta^{-\alpha-2}z^2, & 0 \leq z < \delta, \end{cases}$$

with some $\delta \in (0, 1)$ and

$$(2.4) \quad \alpha = \frac{2}{p-1}.$$

it is easy to check that

$$(2.5) \quad \partial_z^2 \phi + \frac{n-1}{z} \partial_z \phi + \alpha n \phi^p \geq 0$$

holds for all $z \in (0, 1)$.

Take

$$v_0 = \gamma \phi$$

as the initial data of Eq. (2.2), where $\gamma > 0$ is some (random) number. This special choice of initial data is inspired by the deterministic work of [8], [16] and [15]. Since the initial data is isotropic in the space, then the solution $v(x, t)$ is also spatially isotropic for all $t > 0$. Hence, we denote the solution by $v(z, t)$ and Eq. (2.2) can be rewritten as

$$(2.6) \quad \begin{cases} \partial_t v = \partial_z^2 v + \frac{n-1}{z} \partial_z v + K(t) v^p & \text{in } B_1(0), \\ d\xi = \left(-\xi + e^{-rt} \frac{v^r}{\xi^s} \right) dt + \xi dB_t, \\ \frac{\partial v}{\partial z} = 0 & \text{on } \{z = 1\}, \\ v(0) = v_0, \\ \xi(0) = \xi_0, \end{cases}$$

By a Banach fixed point argument as in [27], Eq. (2.6) has a unique *local* solution. The next lemma is about the property of the solution.

Lemma 2.1. *Let v be the solution to Eq. (2.2) on $[0, T]$. Then the following statements hold:*

- (i). $v(t) \geq \gamma$ for all $0 \leq t \leq T$.
- (ii). $\partial_z v(z, t) \leq 0$ for all $0 \leq t \leq T$ and all $0 < z < 1$.
- (iii). For all $\beta \in (0, 1]$, we have $z^n v^\beta(z, t) \leq \overline{v^\beta}(t)$ for all $0 \leq t \leq T$ and all $0 < z < 1$.
- (iv). $\partial_z v(\frac{1}{2}, t) \leq -C_0 2^{n-1}$ for all $0 \leq t \leq T$, where $C_0 > 0$ depends on γ .

Proof. The proofs of (i) and (ii) are the same as those in [16, Lemma 2.1]. By (ii), it is easy to see

$$\begin{aligned} v^\beta(z, t)z^n &= v^\beta(z, t) \int_0^z nr^{n-1}dr \leq \int_0^z v^\beta(r, t)nr^{n-1}dr \\ &\leq \int_0^1 v^\beta(r, t)nr^{n-1}dr = \frac{1}{|B_1(0)|} \int_{B_1(0)} v^\beta(x, t)dx. \end{aligned}$$

Hence, (iii) is proved.

Now we consider $f(z, t) = z^{n-1}\partial_z v$, it is straightforward to check that

$$(2.7) \quad \partial_t f = \partial_z^2 f - \frac{n-1}{z} \partial_z f + pK(t)v^{p-1}f \quad \text{in } B_1(0) \times (0, T).$$

It is easy to check that $f(1, t) = z^{n-1}\partial_z v(z, t)|_{z=1} = 0$ and that $f(z, 0) < -\gamma\alpha$ for all $\frac{1}{4} < z < 1$. Applying strong maximum principle to f , we get $f(\frac{1}{2}, t) \leq -C_0$ for all $t \in (0, T)$, this immediate gives (iv). \square

Define

$$\hat{\xi}(t) = e^{\frac{3t}{2}-B_t}\xi(t) \quad t > 0,$$

we have the following lemma:

Lemma 2.2. *We have*

$$(2.8) \quad \hat{\xi}(t) \geq \hat{\xi}(s) \quad t \geq s \geq 0.$$

Proof. By Itô formula, we have

$$\begin{aligned} d\hat{\xi}(t) &= d\left(e^{\frac{3t}{2}-B_t}\xi(t)\right) \\ &= \xi(t)d\left(e^{\frac{3t}{2}-B_t}\right) + e^{\frac{3t}{2}-B_t}d\xi(t) + \left(de^{\frac{3t}{2}-B_t}\right)\left(d\xi(t)\right) \\ (2.9) \quad &= \xi(t)\left[\frac{3}{2}e^{\frac{3t}{2}-B_t}dt - e^{\frac{3t}{2}-B_t}dB_t + \frac{1}{2}e^{\frac{3t}{2}-B_t}dt\right] \\ &\quad + e^{\frac{3t}{2}-B_t}\left[-\xi(t)dt + e^{-rt}\frac{\overline{v^r}(t)}{\xi^s(t)}dt + \xi(t)dB_t\right] - e^{\frac{3t}{2}-B_t}\xi(t)dt \\ &= e^{-rt+\frac{3t}{2}-B_t}\frac{\overline{v^r}(t)}{\xi^s(t)}dt. \end{aligned}$$

Since $\overline{v^r}(t) \geq 0$ and $\xi(t) \geq 0$ for all $t \geq 0$, $\hat{\xi}(t)$ is an increasing function with respect to t . This completes the proof. \square

Since $\hat{\xi}(0) = \xi_0$, by Lemma 2.2 we have $\hat{\xi}(t) \geq \xi_0$ for all $t \geq 0$. For any $\lambda \in (1, \infty)$, define

$$(2.10) \quad t_\lambda = \inf\{t \geq 0 : \hat{\xi}(t) \geq \lambda\xi_0\}$$

with the convention $\inf \emptyset = \infty$. (t_λ is actually a stopping time). It is easy to see that $t_\lambda = \infty$ holds as long as $\hat{\xi}(t) < \lambda \xi_0$ for all $t > 0$. We clearly have

$$(2.11) \quad \xi_0 \leq \hat{\xi}(t) \leq \lambda \xi_0 \quad t \in [0, t_\lambda].$$

In (2.11), we define $\hat{\xi}(\infty) = \lim_{t \rightarrow \infty} \hat{\xi}(t)$ as $t_\lambda = \infty$.

Let $\theta : \Omega \rightarrow (0, \infty)$ be a positive random variable. From (2.3), we clearly have

$$(2.12) \quad B_{\theta(\omega)}^*(\omega) < \infty \quad \forall \omega \in \Omega,$$

for notational simplicity, we shall suppress the variable ω and write it as B_θ^* . Recall the definition of $K(t)$ in (2.1), we have

$$(2.13) \quad (\lambda \xi_0)^{-q} \exp(-(p-1)\theta - qB_\theta^*) \leq K(t) \leq \xi_0^{-q} \exp\left(\frac{3}{2}q\theta + qB_\theta^*\right), \quad t \in [0, \theta].$$

Indeed, it is easy to see that

$$K(t) = \hat{\xi}(t)^{-q} \exp\left(-(p-1)t + \frac{3}{2}qt - qB_t\right)$$

holds. By (2.11), we have

$$(\lambda \xi_0)^{-q} \exp\left(-(p-1)t + \frac{3}{2}qt - qB_t\right) \leq K(t) \leq \xi_0^{-q} \exp\left(-(p-1)t + \frac{3}{2}qt - qB_t\right),$$

which immediately implies the desired (2.13). For the further usage, we denote

$$(2.14) \quad T_b \text{ the blowup time of the solution } v(z, t),$$

$$(2.15) \quad K_\theta = (\lambda \xi_0)^{-q} \exp(-(p-1)\theta - qB_\theta^*).$$

3. POINTWISE BLOW UP AS $t_\lambda \geq \theta$

Let $\theta \in (0, \infty)$ be some strictly positive random variable as in the previous section. Recall the definition of t_λ in (2.10) with $\lambda \in (1, \infty)$ being some fixed number, under the assumption $t_\lambda \geq \theta$, we shall prove the next two theorems, whose proofs also partly give the main idea of our approach. The first theorem gives a upper bound of the blow up time pointwise, while the second claims that the upper bound of the blowup time is larger than θ as $t_\lambda \geq \theta$, which means that the blow up could happen after the time θ .

Note that the quantities below such as τ and T_b are random variables, we should write them as $\tau(\omega)$ and $T_b(\omega)$ more precisely. For notational simplicity, we shall suppress the argument ω in them if no confusions arise.

Theorem 3.1. *Let $\lambda > 1$ and let $\theta \in (0, \infty)$ be some random number. If $t_\lambda \geq \theta$, choose γ such that $\gamma^{p-1}K_\theta > \frac{4n}{p-1}$, then we have*

$$(3.1) \quad T_b \leq \frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)} \left(1 + \frac{\alpha}{2}\right)^{-p+1} < \frac{\delta^2}{2n} \left(1 + \frac{\alpha}{2}\right)^{-p+1}.$$

Proof. By (2.13), we have

$$K(t) \geq K_\theta, \quad t \in [0, \theta].$$

By (2.6) and the above inequality, we have

$$\begin{cases} \partial_t v \geq \partial_z^2 v + \frac{n-1}{z} \partial_z v + K_\theta v^p & \text{in } B_1(0) \times (0, \theta), \\ \partial_z v = 0 & \text{on } \{z = 1\} \times (0, \theta), \\ v(0) = v_0 & \text{in } B_1(0). \end{cases}$$

Now consider another equation

$$(3.2) \quad \begin{cases} \partial_t w = \partial_z^2 w + \frac{n-1}{z} \partial_z w + K_\theta w^p & \text{in } B_1(0) \times (0, \theta), \\ \partial_z w = 0 & \text{on } \{z = 1\} \times (0, \theta), \\ w(0) = v_0 & \text{in } B_1(0). \end{cases}$$

By comparison principle, we have

$$v(z, t) \geq w(z, t), \quad (z, t) \in B_1(0) \times [0, \theta].$$

Write $\rho = \partial_t w - \frac{K_\theta}{2} w^p$, a straightforward calculation gives

$$\begin{aligned} \partial_t \rho &= \Delta \rho + \frac{K_\theta}{2} p(p-1) w^{p-1} |\nabla w|^2 + \frac{K_\theta}{2} p w^{p-1} \Delta w + \frac{K_\theta}{2} p w^{p-1} \partial_t w \\ &\geq \Delta \rho + \frac{K_\theta}{2} p w^{p-1} \Delta w + \frac{K_\theta}{2} p w^{p-1} \partial_t w \\ &= \Delta \rho + \frac{K_\theta}{2} p w^{p-1} (\partial_t w - K_\theta w^p) + \frac{K_\theta}{2} p w^{p-1} \partial_t w \\ &= \Delta \rho + K_\theta p w^{p-1} \rho, \end{aligned}$$

where the second '=' above is by (3.2). It is straightforward to check that for all $z \in B_1(0)$,

$$\begin{aligned} \rho(z, 0) &= \partial_z^2 u_0 + \frac{n-1}{z} \partial_z u_0 + \frac{K_\theta}{2} u_0^p \\ &= \gamma \left[\partial_z^2 \phi(z) + \frac{n-1}{z} \partial_z \phi(z) + \frac{K_\theta}{2} \gamma^{p-1} \phi^p(z) \right]. \end{aligned}$$

Under the condition in the theorem, (2.5) holds and thus the term in the square bracket is positive. Therefore,

$$\rho(z, 0) \geq 0, \quad z \in B_1(0).$$

It is easy to check

$$\partial_z \rho = 0, \quad (z, t) \in \{z = 1\} \times [0, \theta].$$

Hence, the maximum principle gives

$$\rho(z, t) \geq 0, \quad (z, t) \in B_1(0) \times [0, \theta].$$

That is

$$\partial_t w - \frac{K_\theta}{2} w^p \geq 0, \quad (z, t) \in B_1(0) \times [0, \theta].$$

which implies

$$(3.3) \quad w(z, t) \geq \left[\frac{1}{v_0^{-p+1}(z) - \frac{K_\theta(p-1)t}{2}} \right]^{\frac{1}{p-1}}.$$

By the form of $v_0(z) = \gamma\phi(z)$, for every $z \in (0, 1)$ the term on the right hand side (3.3) blows up at $t = \tau(z)$ with

$$\tau(z) := \begin{cases} \frac{2}{K_\theta(p-1)} \gamma^{-p+1} \left[1 + \frac{1 - (\frac{z}{\delta})^2}{2} \alpha \right]^{-p+1} \delta^2 & z \in [0, \delta], \\ \frac{2}{K_\theta(p-1)} \gamma^{-p+1} z^2 & z \in (\delta, 1), \end{cases}$$

where we have used the relation $\alpha(p-1) = 2$ (see (2.4)). It is easy to see that $\tau(z)$ is an increasing function and $\tau(0) = \frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)} \left(1 + \frac{\alpha}{2}\right)^{-p+1}$, thus we get the desired bound for T_b . \square

Corollary 3.2. *Assume that $\theta \leq \theta_0$ a.s. with $\theta_0 > 0$ being some constant and that $\gamma > 0$ is some (sufficiently large) deterministic number, then we have*

$$(3.4) \quad \mathbb{P} \left(T_b \leq \frac{\delta^2}{2n} \left(1 + \frac{\alpha}{2} \right)^{-p+1} \right) \geq 1 - \frac{\sqrt{\theta_0}}{\sqrt{2\pi}} \frac{4}{A_0} e^{-\frac{A_0^2}{2\theta_0}}$$

with $A_0 = \frac{1}{q} \ln \frac{(p-1)\gamma^{p-1}}{4n(\lambda\xi_0)^q} - \frac{p-1}{q}\theta_0$.

Proof. By Theorem 3.1, it suffices to prove that

$$(3.5) \quad \mathbb{P} \left(\gamma^{p-1} K_\theta > \frac{4n}{p-1} \right) \geq 1 - \frac{\sqrt{\theta_0}}{\sqrt{2\pi}} \frac{4}{A_0} e^{-\frac{A_0^2}{2\theta_0}}.$$

Since K_θ is an decreasing function of θ and $\theta \leq \theta_0$ a.s., we have

$$(3.6) \quad \begin{aligned} \mathbb{P} \left(\gamma^{p-1} K_\theta > \frac{4n}{p-1} \right) &\geq \mathbb{P} \left(\gamma^{p-1} K_{\theta_0} > \frac{4n}{p-1} \right) \\ &= \mathbb{P} \left(B_{\theta_0}^* < \frac{1}{q} \ln \frac{(p-1)\gamma^{p-1}}{4n(\lambda\xi_0)^q} - \frac{p-1}{q}\theta_0 \right) \\ &= 1 - \mathbb{P} \left(B_{\theta_0}^* \geq \frac{1}{q} \ln \frac{(p-1)\gamma^{p-1}}{4n(\lambda\xi_0)^q} - \frac{p-1}{q}\theta_0 \right) \\ &\geq 1 - \frac{\sqrt{\theta_0}}{\sqrt{2\pi}} \frac{4}{A_0} e^{-\frac{A_0^2}{2\theta_0}} \end{aligned}$$

with $A_0 = \frac{1}{q} \ln \frac{(p-1)\gamma^{p-1}}{4n(\lambda\xi_0)^q} - \frac{p-1}{q}\theta_0$. \square

Corollary 3.3. *Assume that the conditions in Theorem 3.1 hold. Let $\gamma \rightarrow \infty$ a.s., then we have*

$$T_b \rightarrow 0, \quad a.s..$$

Proof. By Theorem 3.1, we have

$$T_b \leq \frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)} \left(1 + \frac{\alpha}{2}\right)^{-p+1}.$$

As $\gamma \rightarrow \infty$ a.s., we get $\frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)} \left(1 + \frac{\alpha}{2}\right)^{-p+1}$ a.s. and thus $T_b \rightarrow 0$ a.s.. \square

4. GENERAL POINTWISE BLOW UP RESULT

Recall that T_b is the blowup time of $v(z, t)$ and the K_θ is defined in (2.15), in this section, we shall prove the following blow up theorem:

Theorem 4.1. *Let $\lambda > 1$ and let $p \geq r$ and $\frac{p-1}{r} > \frac{2}{n+2}$. We have the following two statements:*

(i) *In the case $t_\lambda \geq 1$, choose $\gamma > 0$ such that $\gamma^{p-1}K_1 > \frac{4n}{p-1}$, we have*

$$(4.1) \quad T_b \leq \frac{2\delta^2}{\gamma^{p-1}K_1(p-1)} \left(1 + \frac{\alpha}{2}\right)^{-p+1}$$

(ii) *In the case $t_\lambda \leq 1$, there exists some $\hat{\theta} \in (0, 1]$ such that as long as $\gamma^{p-1}K_{\hat{\theta}} > \frac{4n}{p-1}$, we have*

$$(4.2) \quad T_b \leq \frac{2\delta^2}{\gamma^{p-1}K_{\hat{\theta}}(p-1)} \left(1 + \frac{\alpha}{2}\right)^{-p+1}.$$

By the same argument as showing Corollary 3.3, we immediately get the following corollary.

Corollary 4.2. *Assume that the conditions in Theorem 4.1 hold. Let $\gamma \rightarrow \infty$ a.s., then we have*

$$T_b \rightarrow 0, \quad a.s..$$

Let $\beta \in (0, 1]$ be some number to be determined later. Denote

$$h(t) = \overline{v^\beta}(t) \quad t > 0.$$

For the further usage, we define

$$(4.3) \quad h_1(t) = \frac{1}{|B_1(0)|} \int_{B_R(0)} v^\beta(z, t) dz,$$

$$(4.4) \quad h_2(t) = \frac{1}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} v^\beta(z, t) dz.$$

where $R \in (0, 1)$ is some number to be determined later. We also define the following stochastic quantity:

(4.5)

$$h_\lambda^* = h(0) + \beta(\lambda - 1)\lambda^{-q}\gamma^{\beta+p-1-r} \exp\left(-pt_\lambda - (s+q+1)\left(\frac{3t_\lambda}{2} + B_{t_\lambda}^*\right)\right) \xi_0^{s-q+1},$$

it will frequently appear in the arguments below. It is easy to see

$$(4.6) \quad h_\lambda^* \leq h(0) + \beta(\lambda - 1)\lambda^{-q}\gamma^{\beta+p-1-r} \xi_0^{s-q+1}.$$

Lemma 4.3. *Let $\lambda > 1$ and $p \geq r$. Assume $t_\lambda < \infty$. Choose $\beta \in (0, 1]$ such that $p + \beta - 1 \geq r$, then we have*

$$h(t_\lambda) \geq h_\lambda^*.$$

Proof. We have

$$(4.7) \quad \begin{aligned} \frac{dh(t)}{dt} &= \frac{\beta}{|B_1(0)|} \int_{B_1(0)} v^{\beta-1}(x, t) \partial_t v(x, t) dx \\ &= \frac{\beta}{|B_1(0)|} \int_{B_1(0)} v^{\beta-1}(x, t) [\Delta v(x, t) + K(t)v^p(x, t)] dx \\ &= \frac{\beta(1-\beta)}{|B_1(0)|} \int_{B_1(0)} v^{\beta-2}(x, t) |\nabla v(x, t)|^2 dx + \beta K(t) \overline{v^{\beta+p-1}}(t) \\ &\geq \beta K(t) \overline{v^{\beta+p-1}}(t) \end{aligned}$$

where the last inequality is by $\beta \in (0, 1]$. Since $p \geq r$ and $\beta \in (0, 1]$ are such that $p + \beta - 1 \geq r$, by Lemma 2.1 (i), we have $v(z, t) \geq \gamma$ for all $t > 0$ and $0 < z < 1$ and thus

$$\overline{v^{\beta+p-1}}(t) \geq \gamma^{\beta+p-1-r} \overline{v^r}(t).$$

Hence,

$$\frac{dh(t)}{dt} \geq \beta \gamma^{\beta+p-1-r} K(t) \overline{v^r}(t)$$

On the other hand, by (2.9), we have

$$\overline{v^r}(t) = e^{rt - \frac{3t}{2} + B_t} \xi^s(t) \frac{d\hat{\xi}(t)}{dt}.$$

Hence, by (2.1), (2.11) and the above relations, we have

$$\begin{aligned}
\frac{dh(t)}{dt} &\geq \beta \gamma^{\beta+p-1-r} K(t) e^{rt-\frac{3t}{2}+Bt} \xi^s(t) \frac{d\hat{\xi}(t)}{dt} \\
&= \beta \gamma^{\beta+p-1-r} \frac{\hat{\xi}^s(t)}{\hat{\xi}^q(t)} e^{(r-p+1)t} e^{(s-q+1)(-\frac{3t}{2}+Bt)} \frac{d\hat{\xi}(t)}{dt} \\
&\geq \beta \gamma^{\beta+p-1-r} \lambda^{-q} \xi_0^{s-q} e^{-pt} e^{-(s+q+1)(\frac{3t}{2}+Bt)} \frac{d\hat{\xi}(t)}{dt} \\
&\geq \beta \gamma^{\beta+p-1-r} \lambda^{-q} \xi_0^{s-q} \exp \left(-pt_\lambda - (s+q+1) \left(\frac{3t_\lambda}{2} + B_{t_\lambda}^* \right) \right) \frac{d\hat{\xi}(t)}{dt}
\end{aligned}$$

for all $t \in [0, t_\lambda]$. By the definition of t_λ and Lemma 2.2, we immediately get the desired inequality. \square

Stimulated from the previous lemma, we define

$$(4.8) \quad \hat{t}_\lambda = \inf \{ t \geq 0 : h(t) \geq h_\lambda^* \},$$

it is clear that

$$\hat{t}_\lambda \leq t_\lambda$$

and

$$(4.9) \quad h(t) \leq h_\lambda^*, \quad t \in [0, \hat{t}_\lambda].$$

Denote $f(z, t) = z^{n-1} \partial_z v(z, t)$, it is easy to check

$$\mathcal{L}f = \partial_t f - \partial_z^2 f + \frac{n-1}{z} \partial_z f - pK(t)v^{p-1}f = 0.$$

The proof of the next lemma has some similarity to that of [6, Lemma 2.2].

Lemma 4.4. *Let $\lambda > 1$. Let $k \in (1, p)$, $\beta \in (0, 1]$ and $\ell \geq \frac{k}{\beta}$. Assume $t_\lambda < \infty$. As $\varepsilon \leq \varepsilon^*$ with*

$$(4.10) \quad \varepsilon^* = \min \left\{ \alpha \left(1 + \frac{\alpha}{2}\right)^{-k} h^\ell(0), \quad 2^{-\ell n + n} C_0 \gamma^{\beta\ell - k}, \quad \frac{(p-k)\gamma^{p+\ell-k}}{2k(\lambda\xi_0)^q} \exp(-(p-1)t_\lambda - qB_{t_\lambda}^*) \right\},$$

we have

$$(4.11) \quad v(z, t) \leq \left(\frac{2h^\ell(t)}{\varepsilon(k-1)} \right)^{\frac{1}{k-1}} z^{-\frac{2}{k-1}}, \quad \forall t \in [0, \hat{t}_\lambda] \quad \forall z \in (0, \frac{1}{2}].$$

Proof. Denote $\eta(z, t) = f(z, t) + \varepsilon z^n \frac{v^k(z, t)}{h^\ell(t)}$ with $f(z, t) = z^{n-1} \partial_z v(z, t)$ and $\varepsilon > 0$ some number to be determined later and $\ell \geq \frac{k}{\beta}$, we prove the lemma in the following three steps.

Step 1: Property of $\eta(z, t)$ By (i), (iii) and (iv) of Lemma 2.1 and the relation $\ell \geq \frac{k}{\beta}$, we further have

$$\begin{aligned}
 \eta\left(\frac{1}{2}, t\right) &= \left(\frac{1}{2}\right)^{n-1} \partial_z v\left(\frac{1}{2}, t\right) + \varepsilon \left(\frac{1}{2}\right)^n \frac{v^k\left(\frac{1}{2}, t\right)}{h^\ell(t)} \\
 &\leq -C_0 + \varepsilon \left(\frac{1}{2}\right)^n \left(\frac{v^\beta\left(\frac{1}{2}, t\right)}{h(t)}\right)^\ell v^{k-\beta\ell}\left(\frac{1}{2}, t\right) \\
 &\leq -C_0 + \varepsilon \left(\frac{1}{2}\right)^n \left(\frac{v^\beta\left(\frac{1}{2}, t\right)}{h(t)}\right)^\ell \gamma^{k-\beta\ell} \\
 &\leq -C_0 + \varepsilon 2^{n\ell-n} \gamma^{k-\beta\ell}
 \end{aligned}
 \tag{4.12}$$

As $t = 0$, for all $z \in (0, \delta)$, by the relation $\alpha + 2 = p\alpha > k\alpha$, we have

$$\begin{aligned}
 \eta(z, 0) &\leq \left[-\alpha \delta^{-\alpha-2} + \varepsilon \left(1 + \frac{\alpha}{2}\right)^k \frac{1}{h^\ell(0)} \delta^{-\alpha k} \right] z^n \\
 &= \left[-\alpha + \varepsilon \left(1 + \frac{\alpha}{2}\right)^k \frac{1}{h^\ell(0)} \delta^{\alpha+2-\alpha k} \right] \delta^{-\alpha-2} z^n \\
 &\leq \left[-\alpha + \varepsilon \left(1 + \frac{\alpha}{2}\right)^k \frac{1}{h^\ell(0)} \right] \delta^{-\alpha-2} z^n.
 \end{aligned}
 \tag{4.13}$$

For all $z \in (\delta, 1)$, by the relation $\alpha + 2 = p\alpha > k\alpha$ again, we have

$$\begin{aligned}
 \eta(z, 0) &\leq \left[-\alpha + \frac{\varepsilon}{h^\ell(0)} z^{\alpha+2-\alpha k} \right] z^{n-\alpha-2} \\
 &\leq \left[-\alpha + \frac{\varepsilon}{h^\ell(0)} \right] z^{n-\alpha-2}.
 \end{aligned}
 \tag{4.14}$$

Hence, collecting (4.12)-(4.14), as long as

$$\varepsilon \leq \min \left\{ \alpha \left(1 + \frac{\alpha}{2}\right)^{-k} h^\ell(0), \quad 2^{-\ell n+n} C_0 \gamma^{\beta\ell-k} \right\},
 \tag{4.15}$$

we have

$$\eta(z, 0) \leq 0, \quad z \in (0, 1),
 \tag{4.16}$$

$$\eta\left(\frac{1}{2}, t\right) \leq 0, \quad t \in (0, \hat{t}_\lambda).
 \tag{4.17}$$

Step 2: Observe

$$\begin{aligned}
\mathcal{L}\eta &= \mathcal{L} \left(\varepsilon z^n \frac{v^k}{h^\ell} \right) \\
&= -2\varepsilon k z^{n-1} \frac{v^{k-1}}{h^\ell} \partial_z v - \varepsilon(p-k) e^{-(p-1)t} \frac{z^n}{\xi^q} \frac{v^{p-1+k}}{h^\ell} - \varepsilon k(k-1) z^n \frac{v^{k-2}}{h^\ell} (\partial_z v)^2 \\
&\quad - \varepsilon \beta (1-\beta) \ell z^n \frac{v^k}{h^{\ell+1}} \overline{v^{\beta-2} |\nabla v|^2} - \varepsilon \ell \beta e^{-(p-1)t} \frac{z^n}{\xi^q} \frac{v^k}{h^{\ell+1}} \overline{v^{\beta+p-1}} \\
&\leq -2\varepsilon k z^{n-1} \frac{v^{k-1}}{h^\ell} \partial_z v - \varepsilon(p-k) e^{-(p-1)t} \frac{z^n}{\xi^q} \frac{v^{p-1+k}}{h^\ell} \\
&= -2\varepsilon k \frac{v^{k-1}}{h^\ell} \eta + \frac{\varepsilon z^n v^k}{h^{2\ell}} \left[2\varepsilon k v^{k-1} - (p-k) e^{-(p-1)t} \frac{h^\ell v^{p-1}}{\xi^q} \right].
\end{aligned}$$

Recall that $v(t) \geq \gamma$ for all $t \geq 0$ from Lemma 2.1 and that $\xi_0 \leq \hat{\xi}(t) \leq \lambda \xi_0$ for all $t \in [0, t_\lambda]$ where $\hat{\xi}(t) = e^{\frac{3t}{2} - Bt} \xi(t)$, we have

$$\begin{aligned}
e^{-(p-1)t} \frac{h^\ell(t) v^{p-k}(t)}{\xi^q(t)} &= e^{-(p-1)t} \frac{h^\ell(t) v^{p-k}(t)}{\hat{\xi}^q(t) e^{-\frac{q}{2}t + qBt}} \geq e^{-(p-1)t} \frac{\gamma^{p+\ell-k}}{\hat{\xi}^q(t) e^{-\frac{q}{2}t + qBt}} \\
&\geq e^{-(p-1)t} \frac{\gamma^{p+\ell-k}}{(\lambda \xi_0)^q e^{-\frac{q}{2}t + qBt}} \geq \frac{e^{-(p-1)t_\lambda - qB_{t_\lambda}^*} \gamma^{p+\ell-k}}{(\lambda \xi_0)^q}, \quad t \in [0, t_\lambda].
\end{aligned}$$

Hence, as long as

$$(4.18) \quad \varepsilon \leq \frac{(p-k) e^{-(p-1)t_\lambda - qB_{t_\lambda}^*} \gamma^{p+\ell-k}}{2k(\lambda \xi_0)^q},$$

we have

$$(4.19) \quad \mathcal{L}\eta \leq -2\varepsilon k \frac{v^{k-1}}{h^\ell} \eta.$$

Step 3: Choose

$$\varepsilon \leq \min \left\{ \alpha \left(1 + \frac{\alpha}{2}\right)^{-k} h^\ell(0), \quad 2^{-\ell n + n} C_0 \gamma^{\beta \ell - k}, \quad \frac{(p-k) \gamma^{p+\ell-k}}{2k(\lambda \xi_0)^q} \exp(-(p-1)t_\lambda - qB_{t_\lambda}^*) \right\},$$

then (4.19), (4.16) and (4.17) all hold. By comparison principle, we get

$$\eta(z, t) \leq 0, \quad \forall 0 < z < 1 \quad \forall 0 \leq t \leq \hat{t}_\lambda.$$

This implies

$$\partial_z v(z, t) \leq -\varepsilon z \frac{v^k(z, t)}{h^\ell(t)},$$

thus

$$(4.20) \quad v(z, t) \leq \left(\frac{2h^\ell(t)}{\varepsilon(k-1)} \right)^{\frac{1}{k-1}} z^{-\frac{2}{k-1}}, \quad \forall t \in [0, \hat{t}_\lambda] \quad \forall z \in (0, \frac{1}{2}].$$

□

Lemma 4.5. Assume $t_\lambda \leq 1$. For $R \in (0, 1)$, we have

$$(4.21) \quad |\partial_z v(z, t)| \leq C_1, \quad \forall z \in [R, 1] \quad \forall t \in [0, \hat{t}_\lambda],$$

where C_1 is some number depending on R and γ .

Remark 4.6. In the lemma, we assume $t_\lambda \leq 1$, the 1 here can be replaced by any other positive number. It seems that the assumption $t_\lambda \leq 1$ is necessary for getting the bound C_1 which only depends on R .

Proof. Writing $w(z, t) = \partial_z v(z, t)$, by Eq. (2.6) we have

$$(4.22) \quad \partial_t w = \partial_z^2 w + \frac{n-1}{z} \partial_z w + \left(pK(t)v^{p-1} - \frac{n-1}{z^2} \right) w.$$

By (iii) of Lemma 2.1 and (4.9), we have

$$v^\beta(z, t) \leq z^{-n} h(t) \leq z^{-n} h_\lambda^*, \quad \forall t \in [0, \hat{t}_\lambda].$$

This and (4.6) further implies

$$(4.23) \quad v^\beta(z, t) \leq R^{-n} [h(0) + \beta(\lambda - 1)\gamma^{\beta+p-1-r}\lambda^{-q}\xi_0^{s-q+1}], \quad \forall t \in [0, \hat{t}_\lambda], \quad z \in [R, 1].$$

Since $\hat{t}_\lambda \leq t_\lambda \leq 1$, by (2.13), we have

$$(4.24) \quad K(t) \leq \xi_0^{-q} \exp\left(\frac{3}{2}q + qB_1^*\right), \quad t \in [0, \hat{t}_\lambda].$$

We can extend Eq. (4.22) from the time interval $[0, \hat{t}_\lambda]$ to $[0, 1]$ by

$$K(t)v^{p-1}(z, t) = K(\hat{t}_\lambda)v^{p-1}(z, \hat{t}_\lambda), \quad \forall z \in [R, 1] \quad \forall t \in [\hat{t}_\lambda, 1].$$

Now Eq. (4.22) with $(z, t) \in [R, 1] \times [0, 1]$ have uniformly bounded coefficients.

On the other hand, as $t = 0$, it is easy to check

$$|\partial_z \phi(z)| = \alpha z^{-\alpha-1}, \quad z \in [\delta, 1],$$

$$|\partial_z \phi(z)| = \alpha \delta^{-\alpha-2} z, \quad z \in [0, \delta].$$

Indeed, if $R \geq \delta$, then the first identity above implies

$$(4.25) \quad |\partial_z v_0(z)| \leq \alpha \gamma R^{-\alpha-1}, \quad z \in [R, 1];$$

if $R < \delta$, then the second identity above implies (4.25) as well. Hence,

$$|\partial_z v_0(z)| \leq \alpha \gamma R^{-\alpha-1}, \quad z \in [0, 1].$$

So, by parabolic regularity ([17]), we immediately get the desired inequality. □

Lemma 4.7. Assume $t_\lambda \leq 1$. Let $p \geq r$ and $\frac{p-1}{r} > \frac{2}{n+2}$. Let $\beta \in (0, 1]$ be such that $p + \beta - 1 \geq r$ holds. For any $R \in (0, 1)$, we have

$$(4.26) \quad \hat{t}_\lambda \geq \frac{h_\lambda^* - h(0) - n \left(\frac{2(h_\lambda^*)^\ell}{\varepsilon^*(k-1)} \right)^{\frac{1}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}}}{L(\beta, C_1, \lambda, \gamma, p, q, R)},$$

where $k \in (1, p)$, $\ell \geq \frac{k}{\beta}$, ε^* is defined by (4.10), and

$$\begin{aligned} & L(\beta, C_1, \lambda, \gamma, p, q, R) \\ &:= C_1 n \beta R^{n-1} \gamma^{\beta-1} + C_1^2 \beta (1-\beta) \gamma^{\beta-2} + \beta \xi_0^{-q} \exp \left(\frac{3}{2} q t_\lambda + q B_{t_\lambda}^* \right) \left(\frac{h_\lambda^*}{R^n} \right)^{\frac{\beta+p-1}{\beta}} \end{aligned}$$

with C_1 being the number in Lemma 4.5 (which depends on R).

Remark 4.8. We can tune the number R such that the right hand of (4.26) is strictly large than 0 and make the claim $\hat{t}_\lambda > \hat{\theta} > 0$ be true.

Proof. Recall that

$$h(t) = h_1(t) + h_2(t)$$

where

$$h_1(t) = \frac{1}{|B_1(0)|} \int_{B_R(0)} v^\beta(x, t) dx, \quad h_2(t) = \frac{1}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} v^\beta(x, t) dx.$$

with R being some number to be chosen. By Lemma 4.4, we have

$$(4.27) \quad v(z, t) \leq \left(\frac{2h^\ell(t)}{\varepsilon^*(k-1)} \right)^{\frac{1}{k-1}} z^{-\frac{2}{k-1}}, \quad \forall t \in [0, \hat{t}_\lambda] \quad \forall z \in (0, \frac{1}{2}].$$

Since $\frac{p-1}{r} > \frac{2}{n+2}$, we have

$$n(p-1) > 2(r+1-p).$$

Thanks to the condition $p + \beta - 1 \geq r$ with $\beta \in (0, 1]$, we can choose some $\beta \in (0, 1]$ so that

$$n(p-1) > 2\beta \geq 2(r+1-p).$$

Therefore, we can choose some $k \in (1, p)$ so that

$$n(k-1) > 2\beta.$$

Hence, for any $t \in [0, \hat{t}_\lambda]$, by (4.27) and (4.9), we have

$$\begin{aligned}
 h_1(t) &= n \int_0^R v^\beta(z, t) z^{n-1} dz \\
 &\leq n \left(\frac{2h^\ell(t)}{\varepsilon^*(k-1)} \right)^{\frac{\beta}{k-1}} \int_0^R z^{n-1-\frac{2\beta}{k-1}} dz \\
 (4.28) \quad &= n \left(\frac{2h^\ell(t)}{\varepsilon^*(k-1)} \right)^{\frac{\beta}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}} \\
 &\leq n \left(\frac{2(h_\lambda^*)^\ell}{\varepsilon^*(k-1)} \right)^{\frac{\beta}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}}.
 \end{aligned}$$

Now we consider $h_2(t)$, by (2.2), it is easy to see

$$\begin{aligned}
 \frac{d}{dt} h_2(t) &= \frac{d}{dt} \frac{1}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} v^\beta(x, t) dx \\
 &= \frac{\beta}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} v^{\beta-1} (\Delta v + K(t)v^p) dx \\
 &= -n\beta R^{n-1} v^{\beta-1}(R, t) v_z(R, t) - \frac{\beta(\beta-1)}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} v^{\beta-2} |\nabla v|^2 dx \\
 &\quad + \beta K(t) \frac{1}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} v^{\beta+p-1} dx.
 \end{aligned}$$

By (i) of Lemma 2.1 and Lemma 4.5, we have $v^{\beta-1} \leq \gamma^{\beta-1}$, $v^{\beta-2} \leq \gamma^{\beta-2}$ and

$$\begin{aligned}
 \left| \frac{d}{dt} h_2(t) \right| &\leq n\beta R^{n-1} \gamma^{\beta-1}(R, t) |v_z(R, t)| + \frac{\beta(1-\beta)}{|B_1(0)|} \gamma^{\beta-2} \int_{B_1(0) \setminus B_R(0)} |\nabla v|^2 dx \\
 &\quad + \beta \frac{e^{-(p-1)t}}{\xi^q(t)} \frac{1}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} v^{\beta+p-1} dx.
 \end{aligned}$$

By (iii) of Lemma 2.1, we have $v^\beta(z, t) \leq \frac{h(t)}{z^n}$. This and (4.9) further give

$$\begin{aligned}
 \left| \frac{d}{dt} h_2(t) \right| &\leq C_1 n \beta R^{n-1} \gamma^{\beta-1} + C_1^2 \beta (1-\beta) \gamma^{\beta-2} \\
 &\quad + \beta e^{-(p-1)t} \left(\hat{\xi}(t) e^{-\frac{3}{2}t+B_t} \right)^{-q} \frac{1}{|B_1(0)|} \int_{B_1(0) \setminus B_R(0)} \left(\frac{h(t)}{z^n} \right)^{\frac{\beta+p-1}{\beta}} dx \\
 (4.29) \quad &\leq C_1 n \beta R^{n-1} \gamma^{\beta-1} + C_1^2 \beta (1-\beta) \gamma^{\beta-2} \\
 &\quad + \beta \xi_0^{-q} \exp \left(\frac{3}{2} q t_\lambda + q B_{t_\lambda}^* \right) \left(\frac{h_\lambda^*}{R^n} \right)^{\frac{\beta+p-1}{\beta}} \\
 &:= L(\beta, C_1, \lambda, \gamma, p, q, R)
 \end{aligned}$$

for all $t \in [0, \hat{t}_\lambda]$.

By the definition of \hat{t}_λ , (4.28) and (4.29), we have

$$\begin{aligned}
h_\lambda^* - h(0) &\leq h(\hat{t}_\lambda) - h(0) \\
&\leq h_1(\hat{t}_\lambda) + h_2(\hat{t}_\lambda) - h_2(0) \\
&\leq n \left(\frac{2(h_\lambda^*)^\ell}{\varepsilon^*(k-1)} \right)^{\frac{1}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}} + \int_0^{\hat{t}_\lambda} \left| \frac{d}{ds} h_2(s) \right| ds \\
&\leq n \left(\frac{2(h_\lambda^*)^\ell}{\varepsilon^*(k-1)} \right)^{\frac{1}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}} + \hat{t}_\lambda L(\beta, C_1, \lambda, \gamma, p, q, R).
\end{aligned}$$

This immediately implies the desired inequality. \square

Proof of Theorem 4.1. To prove the theorem, we shall consider the two cases: (i) the case $t_\lambda \geq 1$ and (ii) the case $t_\lambda < 1$.

(i) $t_\lambda \geq 1$. Take $\theta = 1$ in Section 3, we immediately get the desired estimate by Theorem 3.1.

(ii) $t_\lambda < 1$. By (4.5), it is easy to see that if $t_\lambda < 1$ we have

$$(4.30) \quad h_\lambda^* \geq h(0) + \beta(\lambda - 1)\lambda^{-q}\gamma^{p+\beta-1-r} \exp \left(-p - (s + q + 1) \left(\frac{3}{2} + B_1^* \right) \right) \xi_0^{s-q+1}.$$

Recalling (4.6) as below:

$$(4.31) \quad h_\lambda^* \leq h(0) + \beta(\lambda - 1)\gamma^{p+\beta-1-r}\lambda^{-q}\xi_0^{s-q+1}.$$

The estimate (4.26), together with (4.30) and (4.31), implies that there exists some $R \in (0, 1)$ (which can be tuned according to $p, q, \lambda, B_1^*, s, \lambda, \gamma, \beta, \xi_0$) and some $\hat{\theta}$ (depending on $\beta, p, q, \lambda, \gamma, B_1^*, s, R$) such that

$$\hat{t}_\lambda \geq \hat{\theta} > 0.$$

$\hat{\theta} \in (0, 1)$ is obvious. Since $t_\lambda \geq \hat{t}_\lambda$, we have $t_\lambda \geq \hat{\theta}$. Now we can use Theorem 3.1 to get the desired result. \square

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